to this substance, it is accompanied by several other bases, the study of which is not yet completed. Nor am I at present in a position to offer any definite opinion regarding the constitution of the new compounds, tempting though it appears to venture on speculations. It is in the hope of rendering the formulæ of the new bases more transparent that I have commenced to examine some of the products of decomposition. Their study is likewise far from being completed; but I may mention, even now, that both rosaniline and leucaniline, when in nitric solution, are powerfully acted upon by nitrous acid, new bases being thus generated, the platinum-salts of which are remarkable for their fulminating properties. A splendid crystalline base also deserves to be mentioned, which, associated with aniline, appears among the products of distillation of rosaniline.

The results obtained in the further prosecution of these studies I propose to lay before the Royal Society in a future communication.

II. "On the Integration of Simultaneous Differential Equations." By GEORGE BOOLE, Esq. Received March 4, 1862.

It is well known that a system of $n-1$ simultaneous differential equations of the first order connecting n variables always admits of $n-1$ integrals, each of which is the form $P=c$, i. e. each of them is expressible by a function of the variables equated to an arbitrary constant.

But when the number of the variables exceeds by more than by unity the number of the differential equations, no existing theory assigns the number of theoretically possible integrals, or guides us to their discovery.

Yet cases such as this occur in problems of the greatest importance. The solution of partial differential equations of the second order by Monge's method depends ultimately on the solution of a system of *three* ordinary differential equations of the first order between *five* variables.

I wish here briefly to indicate the results of a theory which enables us in all such cases, 1st, to assign *à priori* the number of possible integrals; 2ndly, to reduce the determination of the integrals to the solution of a system of differential equations equal in number to the

number of the integrals, and capable of expression in the form of exact differentials.

I will confine my observations to the case of $n-2$ differential equations connecting n variables. The general theory will be seen in the particular one.

1. The solution of $n-2$ differential equations of the first order connecting n variables may be reduced to the solution of a system of 2 linear partial differential equations. To deduce these, let $\mathrm{P}=c$ be any integral of the given system, and suppose $x_1, x_2 \ldots x_n$ the variables, then from

$$\frac{d\mathrm{P}}{dx_1}dx_1+\frac{d\mathrm{P}}{dx_2}dx_2\ldots+\frac{d\mathrm{P}}{dx_n}dx_n=0$$

eliminate by means of the given system $n-2$ of the differentials, and equate to 0 the coefficients of the two remaining and independent ones.

2. Let the two partial differential equations thus formed be

$$\mathrm{A}_1\frac{d\mathrm{P}}{dx_1}+\mathrm{A}_2\frac{d\mathrm{P}}{dx_2}\ldots+\mathrm{A}_n\frac{d\mathrm{P}}{dx_n}=0\ldots\ldots\ldots\ldots \quad (\mathrm{I.})$$

$$\mathrm{B}_1\frac{d\mathrm{P}}{dx_1}+\mathrm{B}_2\frac{d\mathrm{P}}{dx_2}\ldots+\mathrm{B}_n\frac{d\mathrm{P}}{dx_n}=0\,;\ldots\ldots\ldots\ldots \quad (\mathrm{II.})$$

then representing

$$\mathrm{A}_1\frac{d}{dx_1}+\mathrm{A}_2\frac{d}{dx_2}\ldots+\mathrm{A}_n\frac{d}{dx_n}\text{ by }\Delta_1,$$

$$\mathrm{B}_1\frac{d}{dx_1}+\mathrm{B}_2\frac{d}{dx_2}\ldots+\mathrm{B}_n\frac{d}{dx_n}\text{ by }\Delta_2,$$

the equations become

$$\Delta_1\,\mathrm{P}=0,\qquad \Delta_2\,\mathrm{P}=0.\ \ldots\ldots\ldots\ldots\ldots \quad (1)$$

Form now the equation $\Delta_1\,\Delta_2\,\mathrm{P}-\Delta_2\,\Delta_1\,\mathrm{P}=0$, or as it is permitted to express it,

$$(\Delta_1\,\Delta_2-\Delta_2\,\Delta_1)\,\mathrm{P}=0.\ \ldots\ldots\ldots\ldots\ldots\ldots \quad (2)$$

This will also prove a linear partial differential equation of the first order; and if from it by means of (I.) and (II.) we eliminate $\frac{d\mathrm{P}}{dx_{n-1}}$ and $\frac{d\mathrm{P}}{dx_n}$, we shall obtain an equation of the form

$$\mathrm{C}_1\frac{d\mathrm{P}}{dx_1}+\mathrm{C}_2\frac{d\mathrm{P}}{dx_2}\ldots+\mathrm{C}_{n-2}\frac{d\mathrm{P}}{dx_{n-2}}=0\ldots\ldots\ldots \quad (\mathrm{III.})$$

This we shall represent by $\Delta_3\,\mathrm{P}=0$. The equations (I.) and (II.)

may be so prepared as to lead to this equation *directly*. To effect this, it suffices to eliminate from one of these equations $\frac{dP}{dx_n}$, from the other $\frac{dP}{dx_{n-1}}$, and to reduce in each the coefficient of the one which remains to unity, and then apply the theorem (2).

3. Between (I.) and (III.) and between (II.) and (III.) the same process may be applied as between (I.) and (II.). The effect of this is to give new partial differential equations; in fact, to generate a system which will be *complete* when the further application of the method gives rise to no new equations, but only to identities, or to repetitions, or combinations of the equations already obtained. And though any equation of the system may be combined with any other, according to the theorem, in order to form a new one, yet it may be shown that the system will be complete when no new equation arises from the combination of any with the original ones (I.), (II.).

4. Suppose that in this way m partial differential equations have been obtained, including those two into which the given system of ordinary differential equtions was transformed. Then that system of ordinary differential equations will admit of exactly $n-m$ integrals, *i. e.* the number of integrals will be equal to the number of the variables diminished by the number of partial differential equations.

5. To determine these integrals, let the complete system of partial differential equations be represented by

$$\Delta_1 P=0,\ \Delta_2 P=0,\ldots \Delta_m P=0;$$

then multiplying the second by λ_2, the third by λ_3, &c., and adding, we have

$$\Delta_1 P+\lambda_2 \Delta_2 P \ldots +\lambda_m \Delta_m P=0,$$

a single partial differential equation, which, $\lambda_2 \lambda_3 \ldots \lambda_m$ being regarded as indeterminate, will be equivalent to the *system* of equations from which it is formed. Represent this equation by

$$X_1 \frac{dP}{dx_1}+X_2 \frac{dP}{dx_2}\ldots +X_n \frac{dP}{dx_n}=0,$$

then its auxiliary system of ordinary differential equations will be

$$\frac{dx_1}{X_1}=\frac{dx_2}{X_2}\ldots=\frac{dx_n}{X_n}.$$

If from these $n-1$ equations we eliminate the $m-1$ quantities $\lambda_2 \lambda_3 \ldots \lambda_m$, we shall obtain $n-m$ differential equations. *These wil be capable of expression as exact differential equations, and wil give by integration the* n—m *integrals before mentioned.*

The method above described admits of important applications. It enables us to assign beforehand the conditions of success in the application of Monge's and of similar methods to the integration of partial differential equations of the second order, and even to determine the nature of the theoretically possible integral where its actual exhibition in a finite form is impossible. It also enables us to investigate by a new and perfectly rigorous method the conditions of integrability of ordinary differential expressions.

I subjoin a single result of the former of these applications. It is known that the equations of the possible envelopes of any surface

$$z=\phi(x, y, a, b, c),$$

in which three parameters, a, b, c, vary in subjection to two conditions,

$$f_1(a, b, c)=0, \qquad f_2(a, b, c)=0,$$

will satisfy a partial differential equation of the form

$$\mathrm{R}r+\mathrm{S}s+\mathrm{T}t+s^2-rt=\mathrm{V}.$$

The application of the above method shows that, in order that this equation may admit of an integral of the above species, *i. e.* an integral interpretable by the envelope of a surface in which three parameters vary in subjection to two connecting relations, the following conditions are necessary and sufficient, viz.

$$\mathrm{S}^2+4\mathrm{RT}-4\mathrm{V}=0, \quad \ldots\ldots\ldots\ldots (1)$$

$$\Delta\mathrm{R}+\Delta' m=0, \quad \ldots\ldots\ldots\ldots (2)$$

$$\Delta m+\Delta'\mathrm{T}=0, \quad \ldots\ldots\ldots\ldots (3)$$

in which m is one of the equal roots of

$$m^2-\mathrm{S}m+\mathrm{RT}-\mathrm{V}=0,$$

and

$$\Delta=\frac{d}{dx}+p\frac{d}{dz}-m\frac{d}{dq}+\mathrm{T}\frac{d}{dp},$$

$$\Delta'=\frac{d}{dy}+q\frac{d}{dz}-m\frac{d}{dp}+\mathrm{R}\frac{d}{dq}.$$

The first only of the above three conditions appears to have been assigned before (Ampère, Journal de l'École Polytechnique, Cahier xviii.).